ONE-DIMENSIONAL PERTURBATIONS OF UNBOUNDED SELFADJOINT OPERATORS WITH EMPTY SPECTRUM

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Abstract. We study spectral properties of one-dimensional singular perturbations of an unbounded selfadjoint operator and give criteria for the possibility to remove the whole spectrum by a perturbation of this type. A counterpart of our results for the case of bounded operators provides a complete description of compact selfadjoint operators whose rank one perturbation is a Volterra operator.

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1. Introduction

We study *singular* rank one perturbations of an unbounded selfadjoint operator. This paper is a continuation of [5], where the completeness of eigenvectors of these perturbations was considered.

Let μ be a *singular* measure on $\mathbb{R}$ and let $\mathcal{A}$ be the operator of multiplication by the independent variable x in $L^2(\mu)$ (thus, $\mathcal{A}$ is a cyclic *singular* selfadjoint operator). Moreover, we assume that $0 \notin \operatorname{supp}\mu$, and so $\mathcal{A}^{-1}$ is a bounded operator in $L^2(\mu)$.

Now we define *singular rank one perturbations* of $\mathcal{A}$. Let a, b be functions such that

$$\frac{a}{x},\ \frac{b}{x} \in L^2(\mu), \tag{1.1}$$

however, possibly, $a, b \notin L^2(\mu)$. Let $\varkappa \in \mathbb{C}$ be a constant such that

$$\varkappa \neq \int_{\mathbb{R}} x^{-1} a(x)\overline{b(x)}\, d\mu(x) \quad \text{in the case when } a \in L^2(\mu). \tag{1.2}$$

We associate with any such data $(a, b, \varkappa)$ a linear operator $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$, defined as follows:

$$\begin{aligned} \mathcal{D}(\mathcal{L}) &\stackrel{\text{def}}{=} \big\{ y = y_0 + c\cdot\mathcal{A}^{-1}a : \\ &\qquad\qquad c\in\mathbb{C},\ y_0 \in \mathcal{D}(\mathcal{A}),\ \varkappa c + \langle y_0, b\rangle = 0\big\}; \\ \mathcal{L}y &\stackrel{\text{def}}{=} \mathcal{A}y_0, \quad y \in \mathcal{D}(\mathcal{L}). \end{aligned} \tag{1.3}$$

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Condition (1.2) is equivalent to the uniqueness of the decomposition $y = y_0 + c \cdot \mathcal{A}^{-1}a$ in the above formula for $\mathcal{D}(\mathcal{L})$, hence the operator $\mathcal{L}$ is correctly defined. The operator $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ is said to be a *singular rank one perturbation* of $\mathcal{A}$.

Singular perturbations of selfadjoint operators have been studied for a long time, see, for instance, [27, 2, 34].

Essentially, singular rank one perturbations are unbounded algebraic inverses to bounded rank one perturbations of bounded selfadjoint operators. Namely, if the triple $(a, b, \varkappa)$ satisfies (1.2) and $\varkappa \neq 0$, then the bounded operator $\mathcal{A}^{-1} - \varkappa^{-1}\mathcal{A}^{-1}a\,(\mathcal{A}^{-1}b)^*$ has trivial kernel, and

$$\mathcal{L}(\mathcal{A}, a, b, \varkappa) = \big(\mathcal{A}^{-1} - \varkappa^{-1}\mathcal{A}^{-1}a\,(\mathcal{A}^{-1}b)^*\big)^{-1}.$$

Here we denote by $\mathcal{A}^{-1}a\,(\mathcal{A}^{-1}b)^*$ the bounded rank one operator $\mathcal{A}^{-1}a\,(\mathcal{A}^{-1}b)^*f = (f, \mathcal{A}^{-1}b)\mathcal{A}^{-1}a$, $f \in L^2(\mu)$. Conversely, if $\mathcal{A}_0$ is a bounded selfadjoint operator with trivial kernel and $\mathcal{L}_0 = \mathcal{A}_0 + a_0 b_0^*$ is its rank one perturbation and $\operatorname{Ker}\mathcal{L}_0 = 0$, then the algebraic inverse $\mathcal{L}_0^{-1}$ is a singular rank one perturbation of $\mathcal{A}_0^{-1}$. We refer to [5] for details and for similar statements for rank n singular perturbations.

During the last 20 years selfadjoint rank one perturbations of selfadjoint operators were extensively studied by Simon, del Rio, Makarov and many other authors in relation with the problem of stability of the point spectrum and the study of the singular continuous spectrum (see [10] and a survey [37]). Some recent developments can be found in [25, 3]). In what follows, we consider only perturbations of compact selfadjoint operators (or operators with compact resolvent), but the perturbations are no longer selfadjoint. The spectral structure of this class becomes unexpectedly rich and complicated as soon as we leave the classes covered by classical theories (weak perturbations in the sense of Macaev or dissipative operators).

In our preceding paper [5], we studied the completeness of eigenvectors of $\mathcal{L}$ and $\mathcal{L}^*$ as well as the possilbility of the spectral synthesis for such perturbations. Our main tool in [5] was a functional model for rank one singular perturbations. This model realizes singular rank one perturbations as certain 'shift' operators in a so-called model subspace of the Hardy space or in a de Branges space of entire functions.

In this paper, we address the following problem:

Problem 1. *For which measures μ does there exist a singular perturbation $\mathcal{L}$ of $\mathcal{A}$ of the above type whose spectrum is empty?*

Clearly, if such a perturbation exists, then the resolvent of $\mathcal{A}$ is compact, and so the measure μ in question should necessarily be of the form $\mu = \sum_n \mu_n \delta_{t_n}$, where $t_n \in \mathbb{R}$ and $|t_n| \to \infty$, $|n| \to \infty$. Here $\{t_n\}$ may be either one-sided sequence (enumerated by $n \in \mathbb{N}$) or a two-sided sequence (enumerated by $n \in \mathbb{Z}$). Thus, the problem is to describe those spectra $\{t_n\}$ for which the spectrum of the perturbation is empty. Such spectra will be said to be *removable*. It is clear that the property to be removable or nonremovable depends only on $\{t_n\}$, but not on the choice of the masses μ_n.

The change of boundary conditions of an ordinary differential operator leads to a singular one-dimensional perturbation, see, for instance, [5]. This phenomenon of the disappearance of the spectrum if the boundary conditions are properly chosen is well-known, see [32, 20, 6]. As an example, consider the simplest first order selfadjoint operator $\mathcal{A}f(t) = -if'(t)$ on $[0, 2\pi]$ with the boundary condition $f(2\pi) = f(0)$, whose spectrum is $\mathbb{Z}$. The operator $\mathcal{L}f(t) = -if'(t)$ with the changed boundary condition $f(0) = 0$ satisfies

$\mathcal{A} = \mathcal{L}$ on $\mathcal{D}(A) \cap \mathcal{D}(L)$; moreover, $\mathcal{D}(A) \cap \mathcal{D}(L)$ has codimension one both in $\mathcal{D}(A)$ and in $\mathcal{D}(L)$. Therefore $\mathcal{L}$ is a rank one singular perturbation of $\mathcal{A}$ (see [5]). Since the spectrum of $\mathcal{L}$ is empty, the spectrum $\sigma(\mathcal{A}) = \mathbb{Z}$ is removable.

In view of the relation between singular rank one perturbations and usual rank one perturbations of bounded selfadjoint operators, the problem is equivalent to the following:

Problem 2. *Describe those compact selfadjoint operators that have a rank one perturbation which is a Volterra operator.*

Recall that a compact operator is called a Volterra operator if its spectrum equals $\{0\}$ (sometimes the assumption that the kernel is trivial is also included in the definition; all Volterra operators appearing in the present paper have this property). There exist a vast range of results (mainly due to Krein, Gohberg and Macaev) relating the Schatten class properties of the imaginary part of a Volterra operator with the corresponding property for its real part. See [12, Ch. IV] or [13, Ch. III] for these results and for their generalizations to more general symmetric norm ideals; see also a more recent work [31]. Also, in [12, Ch. IV, §10] some partial results are given about the spectra of Volterra operators with finite-dimensional imaginary part. Let us also mention a remarkable theorem, which essentially goes back to Livšic [26] (for an explicit statement see [13, Ch. I, Th. 8.1]): *any dissipative Volterra operator, which is a rank one perturbation of a selfadjoint operator, is unitary equivalent to the integration operator.* In Section 6 we discuss this theorem in more detail and deduce it from out model. Another group of results is concerned with the existence of bases of eigenvectors for rank one perturbations of Volterra integral operators [20, 29, 35].

In the present paper we analyze the situation where the imaginary part of a Volterra operator $\mathcal{L}$ is (at most) a rank two operator and look for the description of the possible spectra of the real part. As far as we know, this particular problem was not previously considered.

Probably, the closest results to ours were obtained by Silva and Tolosa [38] (see also their paper [39] for some more general results) who described entire (in the sense of Krein) operators in terms of spectra of two of their selfadjoint extensions. Their description is based on a theorem due to Woracek [40] characterizing de Branges spaces, which contain zerofree functions. Though our problem deals with only one spectrum, its solution is based on a functional model in a de Branges space and the problem essentially reduces to the existence of a zerofree function in it.

Finite rank perturbations of Volterra operators and their models in de Branges spaces also have been studied in several works by Gubreev and coathors. These papers concern Riesz bases, completeness, generation of C_0 semigroups and the relation with the so-called quasi-exponentials, see [15, 14, 28] and references therein. The paper by Khromov [20] treats spectral properties of finite rank perturbations of Volterra operators from a different point of view; in particular, it contains results stated in terms of the asymptotics of the kernel $M(x,t)$ of an integral Volterra operator near the diagonal.

In this paper, we solve Problems 1 and 2 and obtain a necessary and sufficient condition of removability in terms of entire functions of the so-called Krein class. We say that an entire function F (with $F(0) \neq 0$) is in the *Krein class* $\mathcal{K}_1$, if it is real on $\mathbb{R}$, has only real simple zeros t_n and may be represented as

$$\frac{1}{F(z)} = q + \sum_n c_n \Big(\frac{1}{t_n - z} - \frac{1}{t_n} \Big), \qquad \sum_n t_n^{-2} |c_n| < \infty, \tag{1.4}$$

where $c_n = -1/F'(t_n)$ and $q = 1/F(0)$ (see Section 3 and Lemma 5.1 for details).

Our main result reads as follows:

Theorem 1.1. *Let $t_n \in \mathbb{R}$ and $|t_n| \to \infty$, $|n| \to \infty$. The following are equivalent:*

(*i*) *The spectrum $\{t_n\}$ is removable*;

(*ii*) *There exists a function $F \in \mathcal{K}_1$ such that the zero set of F coincides with $\{t_n\}$.*

An unexpected (and rather counterintuitive) consequence of Theorem 1.1 is that adding a finite number of points to the spectrum helps it to become removable, while deleting a finite number of points may make it nonremovable (see Corollaries 5.2 and 5.3 below).

We have an immediate counterpart of Theorem 1.1 for compact operators which have Volterra rank one perturbations.

Theorem 1.2. *Let $s_n \in \mathbb{R}$, $s_n \neq 0$, and $|s_n| \to 0$, $|n| \to \infty$, and let $\mathcal{A}_0$ be a compact selfadjoint operator with simple point spectrum $\{s_n\}$. The following are equivalent:*

(*i*) *There exists a rank one perturbation $\mathcal{L}_0 = \mathcal{A}_0 + a_0 b_0^*$ such that $\mathcal{L}_0$ is a Volterra operator*;

(*ii*) *The points $t_n = s_n^{-1}$ form the zero set of some function $F \in \mathcal{K}_1$.*

The paper is organized as follows. In Sections 2 and 3 we give some preliminaries on the functional model from [5] and on de Branges' theory. The proofs of Theorems 1.1 and 1.2 are given in Sections 4 and 5. Section 6 contains some examples of removable and nonremovable spectra, while in Section 7 we discuss a simple proof of Livšic's theorem by our methods. In Section 8 we show that sometimes the Volterra property may be achieved by sufficiently 'smooth' perturbations and compare this result with a classical completeness theorem due to Macaev and some results from [5].

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2. Functional model

We use the notations $\mathbb{C}^\pm = \{z \in \mathbb{C} : \pm \mathrm{Im}\, z > 0\}$ for the upper and the lower half-planes and set $H^2 = H^2(\mathbb{C}^+)$. Recall that a function Θ is said to be an *inner function* in $\mathbb{C}^+$ if it is a bounded analytic function with $|\Theta| = 1$ a.e. on $\mathbb{R}$ in the sense of nontangential boundary values. Each inner function Θ generates a *shift-coinvariant* or *model subspace* $K_\Theta \overset{\mathrm{def}}{=} H^2 \ominus \Theta H^2$ of the Hardy space H^2 (we refer to, e.g., [33] for the theory of model spaces and for their numerous applications).

The following functional model of singular rank one perturbations was constructed in [5]. Essentially, it is similar to functional models by Kapustin [19] (for the rank one perturbations of unitary operators) and Gubreev and Tarasenko [14] (for the compact resolvent case); for a more general model see [36]. In what follows we assume that b is a *cyclic* vector for the resolvent of $\mathcal{A}$, so $b \neq 0$ μ-a.e.

Theorem 2.1 ([5], Theorem 0.6)**.** *Let $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ be a singular rank one perturbation of $\mathcal{A}$, and let b be a cyclic vector for $\mathcal{A}^{-1}$. Then there exist an inner function Θ, such that Θ is analytic in a neighborhood of 0, $1 + \Theta \notin H^2$, $\Theta(0) \neq -1$, and a function φ satisfying*

$$\varphi \notin H^2, \qquad \frac{\varphi(z) - \varphi(i)}{z - i} \in K_\Theta, \tag{2.1}$$

such that $\mathcal{L}$ is unitary equivalent to the operator $\mathcal{T} = \mathcal{T}(\Theta, \varphi)$ which acts on the model space $K_\Theta \stackrel{\text{def}}{=} H^2 \ominus \Theta H^2$ by the formulas

$$\mathcal{D}(\mathcal{T}) \stackrel{\text{def}}{=} \{f = f(z) \in K_\Theta : \text{there exists } c = c(f) \in \mathbb{C} : zf - c\varphi \in K_\Theta\},$$

$$\mathcal{T}f \stackrel{\text{def}}{=} zf - c\varphi, \qquad f \in \mathcal{D}(\mathcal{T}).$$

Conversely, any inner function Θ which is analytic in a neighborhood of 0 *and satisfies $1 + \Theta \notin H^2$, $\Theta(0) \neq -1$, and any function φ satisfying* (2.1) *correspond to some singular rank one perturbation $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ of the operator $\mathcal{A}$ of multiplication by the independent variable in $L^2(\mu)$, where μ is some singular measure on $\mathbb{R}$ and $x^{-1}a(x)$, $x^{-1}b(x) \in L^2(\mu)$.*

The functions Θ and φ appearing in the model for $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$ are given by the following formulas. Put

$$\begin{aligned} \beta(z) &= \varkappa + zb^*(\mathcal{A} - z)^{-1}\mathcal{A}^{-1}a \\ &= \varkappa + \int \left(\frac{1}{x - z} - \frac{1}{x}\right) a(x)\overline{b(x)}\, d\mu(x), \end{aligned} \tag{2.2}$$

$$\rho(z) = \delta + zb^*(\mathcal{A} - z)^{-1}\mathcal{A}^{-1}b = \delta + \int \left(\frac{1}{x - z} - \frac{1}{x}\right) |b(x)|^2\, d\mu(x), \tag{2.3}$$

where δ is an arbitrary real constant. Then Θ and φ are defined as

$$\Theta(z) = \frac{i - \rho(z)}{i + \rho(z)}, \qquad \varphi(z) = \frac{\beta(z)}{2}\big(1 + \Theta(z)\big). \tag{2.4}$$

The above functional model uses essentially the properties of the so-called Clark measures introduced in [9]. Recall that the Clark measure σ_ζ, $|\zeta| = 1$, is the measure from the Herglotz representation

$$i\,\frac{\zeta + \Theta(z)}{\zeta - \Theta(z)} = p_\zeta z + q_\zeta + \frac{1}{\pi}\int_{\mathbb{R}} \Big(\frac{1}{t - z} - \frac{t}{t^2 + 1}\Big) d\sigma_\zeta(t), \qquad z \in \mathbb{C}^+,$$

where $p_\zeta \geq 0$, $q_\zeta \in \mathbb{R}$ and $\int_{\mathbb{R}} (1 + t^2)^{-1} d\sigma_\zeta(t) < \infty$. Note that if Θ is meromorphic, then any Clark measure σ_ζ is discrete.

It follows from the results of Ahern and Clark [1] that

$$\zeta - \Theta \in H^2 \iff p_\zeta > 0. \tag{2.5}$$

Note that in our model

$$i\,\frac{1 - \Theta(z)}{1 + \Theta(z)} = \delta + \int \left(\frac{1}{x - z} - \frac{1}{x}\right) |b(x)|^2\, d\mu(x). \tag{2.6}$$

Thus, the measure $\pi|b|^2\mu$ is the Clark measure σ_{-1} for Θ.

Let us mention the following result of [5].

Proposition 2.2 ([5], Proposition 2.2)**.** *Let a, b be functions that satisfy* (1.1) *and let $\varkappa \in \mathbb{R}$. Let Θ and φ be defined by* (2.4)*. Then we have:*

1. *$1 + \Theta \notin H^2$, $\Theta(0) \neq -1$, and $\frac{\varphi}{z+i} \in H^2$;*
2. *If $a \notin L^2(\mu)$, then $\varphi \notin H^2$;*
3. *If $a \in L^2(\mu)$, then $\varphi \in H^2$ if and only if $\varkappa = \sum_n a_n \overline{b}_n t_n^{-1} \mu_n$.*

Since we are interested in the case when μ is a discrete measure: $\mu = \sum_n \mu_n \delta_{t_n}$, where $|t_n| \to \infty$, $|n| \to \infty$, the function Θ is meromorphic in the whole complex plane and analytic on $\mathbb{R}$; so is φ and any element of K_Θ. This situation reduces to the study of de Branges spaces of entire functions (see Section 3 below).

From now on we assume that Θ and φ are meromorphic. By the well-known properties of the model spaces K_Θ, a function $f \in H^2(\mathbb{C}^+)$ is in K_Θ if and only if the function $\tilde{f}(z) = \Theta(z)\overline{f(\overline{z})}$ also is in $H^2(\mathbb{C}^+)$. Analogously, we put

$$\tilde{\varphi}(z) = \Theta(z)\overline{\varphi(\overline{z})}.$$

Denote by Z_φ the set of zeros of φ in $\operatorname{clos}\mathbb{C}^+$ and put $\overline{Z}_{\tilde{\varphi}} = \{z \in \operatorname{clos}\mathbb{C}^- : \overline{z} \in Z_{\tilde{\varphi}}\}$. It follows from [5, Lemma 2.1] that the functions

$$h_\lambda(z) = \frac{\varphi(z)}{z-\lambda}, \qquad \lambda \in Z_\varphi \cup \overline{Z}_{\tilde{\varphi}}$$

belong to K_Θ, and, moreover, all eigenfunctions of the model operator $\mathcal{T}$ are of the form h_λ, $\lambda \in Z_\varphi \cup \overline{Z}_{\tilde{\varphi}}$.

Lemma 2.3 ([5], Lemma 2.4)**.** *Let meromorphic Θ and φ correspond to a singular rank one perturbation of a cyclic selfadjoint operator $\mathcal{A}$ with the compact resolvent. Then the following holds:*

1. *Operators $\mathcal{L}$ and $\mathcal{T}$ have compact resolvents*;
2. $\sigma(\mathcal{T}) = \sigma_p(\mathcal{T}) = Z_\varphi \cup \overline{Z}_{\tilde{\varphi}}$;
3. *The eigenspace of $\mathcal{T}$ corresponding to an eigenvalue $\lambda \in Z_\varphi \cup \overline{Z}_{\tilde{\varphi}}$, is spanned by h_λ.*

3. Preliminaries on entire functions

An entire function E is said to be in the *Hermite–Biehler class* (which we denote by HB) if

$$|E(z)| > |E(\overline{z})|, \qquad z \in \mathbb{C}^+.$$

We also always assume that $E \neq 0$ on $\mathbb{R}$. For a detailed study of the Hermite–Biehler class see [24, Chapter VII]. Put $E^*(z) = \overline{E(\overline{z})}$. If $E \in HB$, then $\Theta = E^*/E$ is an inner function which is meromorphic in the whole plane $\mathbb{C}$; moreover, any meromorphic inner function can be obtained in this way for some $E \in HB$ (see, e.g., [17, Lemma 2.1]).

Given $E \in HB$, we can always write it as $E = A - iB$, where

$$A = \frac{E+E^*}{2}, \qquad B = \frac{E^*-E}{2i}.$$

Then A, B are real on the real axis and have simple real zeros. Moreover, if $\Theta = E^*/E$, then $2A = (1+\Theta)E$.

Any function $E \in HB$ generates the de Branges space $\mathcal{H}(E)$ which consists of all entire functions f such that f/E and f^*/E belong to the Hardy space H^2, and $\|f\|_E = \|f/E\|_{L^2(\mathbb{R})}$ (for the de Branges theory see [8]). It is easy to see that the mapping $f \mapsto f/E$ is a unitary operator from $\mathcal{H}(E)$ onto K_Θ with $\Theta = E^*/E$ (see, e.g., [17, Theorem 2.10]).

An entire function F is said to be of *Cartwright class* if it is of finite exponential type and

$$\int_{\mathbb{R}} \frac{\log^+|F(x)|}{1+x^2}dx < \infty$$

(recall that $\log^+ t = \max(\log t, 0)$, $t > 0$). For the theory of the Cartwright class we refer to [16, 22]. It is well-known that the zeros z_n of a Cartwright class function F have a certain symmetry: in particular,

$$F(z) = Kz^m e^{icz}\, v.p. \prod_n \Big(1 - \frac{z}{z_n}\Big) \stackrel{\text{def}}{=} Kz^m e^{icz} \lim_{R\to\infty} \prod_{|z_n|\le R} \Big(1 - \frac{z}{z_n}\Big), \tag{3.1}$$

where the infinite product converges in the 'principal value' sense, $c \in \mathbb{R}$ and $K \in \mathbb{C}$ are some constants, $m \in \mathbb{Z}_+$.

It follows from this representation that a Cartwright class function is determined uniquely by its zeros, up to a factor $Ke^{i\gamma z}$, where $K \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ are constants.

A function f analytic in $\mathbb{C}^+$ is said to be of *bounded type* if $f = g/h$ for some functions $g, h \in H^\infty(\mathbb{C}^+)$. If, moreover, h can be taken to be outer, we say that f is in *the Smirnov class in* $\mathbb{C}^+$. It is well known that if f is analytic in $\mathbb{C}^+$ and $\operatorname{Im} f > 0$ (such functions are said to be the *Herglotz functions*), then f is in the Smirnov class [16, Part 2, Chapter 1, Section 5]. In particular, if $t_n \in \mathbb{R}$, $u_n > 0$ and $\sum_n u_n < \infty$, then the function $\sum_n \frac{u_n}{t_n - z}$ is in the Smirnov class in $\mathbb{C}^+$. Consequently, $\sum_n \frac{v_n}{t_n - z}$ is in the Smirnov class in $\mathbb{C}^+$ for any $\{v_n\} \in \ell^1$.

The following theorem due to M.G. Krein (see, e.g., [16, Part II, Chapter 1]) will be useful: *If an entire function F is of bounded type both in $\mathbb{C}^+$ and in $\mathbb{C}^-$, then F is of Cartwright class. If, moreover, F is in the Smirnov class both in $\mathbb{C}^+$ and in $\mathbb{C}^-$, then F is a Cartwright class function of zero exponential type.*

We also consider the class of entire functions introduced by M.G. Krein [23] (see also [24, Chapter 6]). Assume that F is an entire function, which is real on $\mathbb{R}$, with simple real zeros $t_n \neq 0$ such that, for some integer $p \geq 0$, we have

$$\sum_n \frac{1}{|t_n|^{p+1} |F'(t_n)|} < \infty$$

and

$$\frac{1}{F(z)} = R(z) + \sum_n \frac{1}{F'(t_n)} \cdot \Big(\frac{1}{z - t_n} + \frac{1}{t_n} + \frac{z}{t_n^2} + \cdots + \frac{z^{p-1}}{t_n^p}\Big), \tag{3.2}$$

where R is some polynomial. The class of such functions F we will denote by $\mathcal{K}_p$. If $F \in \mathcal{K}_p$ for some p, then F is of Cartwright class [24, Chapter 6].

4. First criterion of removability

Recall that the spectrum $\{t_n\}$ with $0 \notin \{t_n\}$ is said to be *removable* if there exists a singular perturbation $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ of $\mathcal{A}$, whose spectrum is empty. Here $\mathcal{A}$ is the operator of multiplication by x in $L^2(\mu)$, $\mu = \sum_n \mu_n \delta_{t_n}$. In this case, given $a, b \in xL^2(\mu)$ we will write a_n and b_n in place of $a(t_n)$ and $b(t_n)$.

It is obvious that if the spectrum of $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ is empty, then b must be a cyclic vector for $\mathcal{A}^{-1}$. In fact, if $b_n = 0$ then the vector e_n defined by $e_n(t_k) = \delta_{nk}$ will be an eigenvector of $\mathcal{L}$ corresponding to the eigenvalue t_n. Indeed, e_n belongs to $\mathcal{D}(\mathcal{L})$ since $(e_n, b) = 0$ and we may take $c = 0$.

Since b is cyclic for $\mathcal{A}^{-1}$, we may apply the functional model from Section 2. Then, in view of Theorem 2.1 and Lemma 2.3, we have an immediate criterion of removability.

Proposition 4.1. *The spectrum $\{t_n\}$ is removable if and only if there exist a meromorphic inner function Θ with $\{t : \Theta(t) = -1\} = \{t_n\}$, $\Theta(0) \neq -1$, $1 + \Theta \notin H^2$, and a function φ which satisfies* (2.1) *such that both φ and $\tilde{\varphi}$ have no zeros in $\mathbb{C}^+ \cup \mathbb{R}$.*

Proof. Indeed, if such pair (Θ, φ) exists, then, by the converse statement in Theorem 2.1 there is a singular measure μ, functions a and b and a constant $\varkappa$ such that $\mathcal{L}(\mathcal{A}, a, b, \varkappa)$ is unitarily equivalent to the model operator $\mathcal{T}(\Theta, \varphi)$. Moreover, in this case Θ and φ are related to the data $(a, b, \varkappa)$ by formulas (2.2)–(2.4). Thus, $|b|^2\mu$ is the Clark measure σ_{-1} for Θ whence $\{t : \Theta(t) = -1\} = \{t_n\}$. By Lemma 2.3 the spectrum of $\mathcal{T}(\Theta, \varphi)$ is empty if and only if φ and $\tilde{\varphi}$ do not vanish in $\mathbb{C}^+ \cup \mathbb{R}$. □

The following statement gives a more palpable description of removable spectra. In particular, we will see that the function φ may be chosen of the form $1/E$ for a function E in the Hermite–Biehler class.

Theorem 4.2. *The spectrum $\{t_n\}$ is removable if and only if the following two conditions hold:*

(*1*) *The set $\{t_n\}$ is the zero set of an entire function in the Cartwright class, and so the* generating function *of the set $\{t_n\}$,*

$$A(z) = v.p. \prod \Big(1 - \frac{z}{t_n}\Big) = \lim_{R \to \infty} \prod_{|t_n| \le R} \Big(1 - \frac{z}{t_n}\Big), \tag{4.1}$$

is well-defined and belongs to the Cartwright class;

(*2*) *Moreover, there exists an entire function E of the Hermite–Biehler class such that $E + E^* = 2A$, $A \notin \mathcal{H}(E)$ and $\frac{1}{(z+i)E} \in H^2$.*

If the spectrum is removable (*so that* (*1*) *and* (*2*) *hold*), *then the pair (Θ, φ), corresponding to a perturbation of $\mathcal{A}$ with empty spectrum and a function E in* (*2*) *may be chosen so that $\Theta = E^*/E$ and $\varphi = 1/E$.*

Proof. Necessity of 1 and 2. If the spectrum $\{t_n\}$ is removable, then there is a pair (Θ, φ) satisfying all conditions in Proposition 4.1. Since φ is a function of bounded type (and even of Smirnov class) which does not vanish in $\mathbb{C}$ and is analytic in a neighborhood of $\mathbb{R}$, its inner-outer factorization (see, e.g., [21]) is of the form

$$\varphi(z) = e^{ic_1 z}\mathcal{O}(z),$$

where $c_1 \ge 0$ and $\mathcal{O}$ is an outer function with $|\mathcal{O}| = |\varphi|$ on $\mathbb{R}$. Since $\tilde{\varphi}$ is also a function of Smirnov class, we have for $t \in \mathbb{R}$,

$$\tilde{\varphi}(t) = \Theta(t)\overline{\mathcal{O}(t)}e^{-ic_1 t} = \mathcal{O}(t)e^{ic_2 t} = \varphi(t)e^{i(c_2 - c_1)t}$$

for some $c_2 \ge 0$. We conclude that for $z \in \mathbb{R}$ and, hence, for any $z \in \mathbb{C}$,

$$e^{2icz} = \frac{\varphi(z)}{\tilde{\varphi}(z)}, \tag{4.2}$$

where $2c = c_1 - c_2$.

The function $1/\varphi$ is a meromorphic function, which has no poles in $\mathbb{C}^+$ and on $\mathbb{R}$. Also, by (4.2),

$$\frac{1}{\overline{\varphi(\overline{z})}} = \frac{\Theta(z)}{\varphi(z)}\, e^{2icz}, \qquad z \in \mathbb{C}^+,$$

and so $1/\varphi$ has no poles in $\mathbb{C}^-$. We conclude that $E = e^{icz}/\varphi$ is an entire function. The function E is in HB, because $E^*/E = \Theta$. Also, E is of bounded type both in the upper and the lower half-planes, and so is of Cartwright class by Krein's theorem.

Now put $\varphi_1 = e^{-icz}\varphi = 1/E$. We assert that we can replace φ with φ_1, that is, that φ_1 also satisfies the conditions of Proposition 4.1. Indeed, we know that

$$\frac{\varphi}{z+i} = \frac{e^{icz}}{(z+i)E} \in H^2$$

(see (2.1)). We assert that $\frac{\varphi_1}{z+i} = e^{-icz}\frac{\varphi}{z+i}$ is also in H^2. If $c \le 0$, then it is obvious. On the other hand, since $E \in HB$ is of order at most one and does not vanish on $\mathbb{R}$, it admits the following factorization (see, e.g., [24, Chapter VII])

$$E(z) = Ke^{-iaz+bz}\prod_n \left(1 - \frac{z}{z_n}\right)e^{h_n z},$$

where $K \in \mathbb{C}$, $a \ge 0$, $b \in \mathbb{R}$, $\{z_n\}$ is a finite or infinite sequence of points in $\mathbb{C}^-$, satisfying the Blaschke condition, and $h_n = \operatorname{Re}\frac{1}{z_n} \ge 0$. It follows that $|E(iy)| \to \infty$ when $y \to \infty$. Hence, when $c > 0$, $\frac{\varphi(iy)}{i(y+1)} = o(e^{-cy})$, and, thus, the function $\frac{\varphi}{z+i}$ is divisible by e^{icz} in H^2. Since $\varphi \notin H^2$, it follows that $\varphi \notin L^2(\mathbb{R})$, so that $\varphi_1 \notin H^2$. Next, since $\tilde{\varphi}_1 = \varphi_1$, it follows that $\Theta\,\frac{\bar{\varphi}_1(t)-\overline{\varphi_1(i)}}{t+i}\Big|_{\mathbb{R}}$ is in H^2, which implies that $\frac{\varphi_1(z)-\overline{\varphi_1(i)}}{z-i} \in K_\Theta$. Hence (2.1) holds.

We get that E is both in the Hermite–Biehler and in the Cartwright class and satisfies $\frac{1}{(z+i)E} = \frac{\varphi_1}{z+i} \in H^2$. Then $A = \frac{E+E^*}{2}$ is a Cartwright class function with zero set $\{t : \Theta(t) = -1\} = \{t_n\}$. Since $1+\Theta \notin H^2$, we conclude that $A = \frac{(1+\Theta)E}{2} \notin \mathcal{H}(E)$.

Notice that this argument also proves the last statement of the Theorem.

Sufficiency of 1 and 2. If E and A are given, we may put $\Theta = E^*/E$ and $\varphi = 1/E$. Then Θ and φ satisfy all conditions of Proposition 4.1, except, may be, one: it may happen that $\varphi = 1/E \in H^2$.

However, the function E such that $A = \frac{E+E^*}{2}$ is not unique. Namely, the function $E_* = A - iB_*$ is in HB if and only if B_*/A is a Herglotz function in $\mathbb{C}^+$, which holds if and only if there exist $\nu_n \ge 0$, $\sum_n t_n^{-2}\nu_n < \infty$, $p \ge 0$ and $r \in \mathbb{R}$ such that

$$\frac{B_*(z)}{A(z)} = pz + r + \sum_n \nu_n\Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big). \tag{4.3}$$

In particular, there exist ν_n^0 and r_0 such that for our initial $E = A - iB$ we have

$$\frac{B(z)}{A(z)} = r_0 + \sum_n \nu_n^0\Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big).$$

Since $1+\Theta \notin H^2$ and $\frac{B}{A} = i\frac{1-\Theta}{1+\Theta}$, the corresponding summand $p_0 z$ is absent by (2.5). Now consider the functions B_* such that

$$\frac{B_*(z)}{A(z)} = r_0 + \sum_n \nu_n\Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big),$$

where ν_n are free parameters; then

$$E_*(z) = A(z)\bigg(1 - ir_0 - i\sum_n \nu_n\Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big)\bigg).$$

Clearly, $E_*(t_n) = i\nu_n A'(t_n)$, and choosing $\{\nu_n\}$ rapidly decreasing we can achieve that $\frac{1}{E_*} \notin L^2(\mathbb{R})$. On the other hand, for the choice of $\nu_n = \nu_n^0$ we have $E_* = E$, and so $\frac{1}{(x+i)E} \in L^2(\mathbb{R})$. Since E_* is a 'continuous function of ν_n', it is not difficult to show that there exist data $\{\nu_n\}$ such that $\frac{1}{(x+i)E_*} \in L^2(\mathbb{R})$, whereas $\frac{1}{E_*} \notin L^2(\mathbb{R})$.

For the convenience of the reader who might be not satisfied with the above 'continuity' argument, we give a rigorous proof of the existence of such sequence $\{\nu_n\}$. It may be assumed that the sequence $\{t_n\}$ has infinitely many positive terms. We will choose a rapidly increasing subsequence $\{t_{n_k}\}_{k=1}^\infty$ of $\{t_n\}$ such that $t_{n_k} \to +\infty$. We will set

$$E_*(z) = A(z)\eta_*(z), \qquad \text{where } \eta_*(z) = 1 - ir_0 - i\sum_n \nu_{n,*}\Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big),$$

with

$$\nu_{n,*} = \begin{cases} \nu_n^0, & n \neq n_k, \\ \nu_k', & n = n_k, \end{cases}$$

where $0 < \nu_k' \le \nu_{n_k}^0$. We will also define auxiliary points τ_k such that $t_{n_{k-1}} \le \tau_{k-1} \le t_{n_k}$ for $k \ge 2$. The sequences $\{n_k\}$, $\{\tau_k\}$ and the weights $\{\nu_k'\}$ will be defined by induction. To do that, we first introduce some more notation. For $k \ge 0$, let

$$\nu_{n,k} = \begin{cases} \nu_n^0, & n \neq n_\ell \text{ or } n = n_\ell,\ \ \ell > k, \\ \nu_\ell', & n = n_\ell,\ \ 1 \le \ell \le k, \end{cases}$$

denote the weight, changed only in the points $t_{n_1}, \dots, t_{n_k}$. Put

$$E_k(z) = A(z)\eta_k(z), \qquad \text{where } \eta_k(z) = 1 - ir_0 - i\sum_n \nu_{n,k}\Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big)$$

(so that $E_0 = E$). It is easy to see that for any such choice, $\frac{1}{(x+i)E_k} \in L^2(\mathbb{R})$ for all k.

The inductive definition is as follows. On the first step, choose any $t_{n_1} > 0$ and any $\tau_1 > \max(4, 2t_{n_1})$. Now suppose that the numbers t_{n_ℓ}, ν_ℓ' and τ_ℓ ($\ell = 1, \dots, k-1$) have been already chosen. On the kth step, n_k, ν_k' and τ_k will be defined. We will use the notation $J_\ell = [-\tau_\ell, \tau_\ell]$.

Choose $t_{n_k} > 2\tau_{k-1}$ so that $\nu_{n_k} t_{n_k}^{-2} \le 2^{-k-1}\tau_{k-1}^{-1}$. It is possible because $\sum_n t_n^{-2}\nu_n < \infty$.

If $\big\|\frac{1}{(x+i)E_{k-1}}\big\|_{L^2(\mathbb{R}\setminus J_{k-1})} \ge 2\tau_{k-1}^{-1}$, then we put $\nu_k' = \nu_{n_k}$ (so that $E_k = E_{k-1}$). Otherwise, we choose $\nu_k' \in (0, \nu_{n_k})$ so that $\big\|\frac{1}{(x+i)E_k}\big\|_{L^2(\mathbb{R}\setminus J_{k-1})} = 2\tau_{k-1}^{-1}$. It is possible because this norm is continuous as a function of ν_k' and tends to infinity as $\nu_k' \to 0^+$. Next, in both cases choose $\tau_k > t_{n_k}$ such that $\big\|\frac{1}{(x+i)E_k}\big\|_{L^2(J_k\setminus J_{k-1})} = \tau_{k-1}^{-1}$. Notice that $\tau_k > 2\tau_{k-1}$, which gives that $\tau_k > 2^{k+1}$.

We claim that the following properties hold.

(i) $|\eta_\ell(x) - \eta_{\ell-1}(x)| \le 2^{-\ell}$ on J_k for $\ell > k$;
(ii) $\big\|\frac{1}{(x+i)E_k}\big\|_{L^2(\mathbb{R})} \le C$ for some constant C, independent of k;
(iii) The sequence of functions $\frac{1}{E_\ell}$ converges uniformly to $\frac{1}{E_*}$ on J_k for any k.

These properties imply our statement. Indeed, (i) gives that for $\ell > m \geq k$,

$$\left|1 - \frac{\eta_m(x)}{\eta_\ell(x)}\right| \leq |\eta_\ell(x) - \eta_m(x)| \leq \sum_{j=m}^{\ell-1} 2^{-j-1} \leq 2^{-m} \qquad \text{for } x \in J_k. \tag{4.4}$$

Next, (ii) and (iii) imply that the functions $\frac{1}{(x+i)E_k}$ converge weakly to $\frac{1}{(x+i)E_*}$ in $L^2(\mathbb{R})$. In particular, $\frac{1}{(x+i)E_*}$ is in $L^2(\mathbb{R})$. Fix some k. For any $\ell > k$, $|\eta_k/\eta_\ell| \geq \frac{1}{2}$ on J_k, and therefore

$$\left\|\frac{1}{E_\ell}\right\|_{L^2(J_k\setminus J_{k-1})} = \left\|\frac{\eta_k}{\eta_\ell}\cdot\frac{1}{E_k}\right\|_{L^2(J_k\setminus J_{k-1})} \geq \frac{\tau_{k-1}}{2}\left\|\frac{1}{(x+i)E_k}\right\|_{L^2(J_k\setminus J_{k-1})} \geq \frac{1}{2},$$

which by (iii) implies that $\|\frac{1}{E_*}\|_{L^2(J_k\setminus J_{k-1})} \geq \frac{1}{2}$ for any k. Therefore $\frac{1}{E_*} \notin L^2(\mathbb{R})$.

So it remains to check (i)–(iii).

Proof of (i): Let $\ell > k$, and let $x \in J_k \subset J_{\ell-1}$. Then

$$|\eta_\ell(x) - \eta_{\ell-1}(x)| = (\nu_{n_\ell} - \nu'_\ell)\frac{|x|}{|t_{n_\ell} - x|t_{n_\ell}} \leq \frac{\nu_{n_\ell}\cdot 2\tau_{\ell-1}}{t_{n_\ell}^2} \leq 2^{-\ell}.$$

In particular, (4.4) holds.

Proof of (ii): For any index k such that $E_k \neq E_{k-1}$, one has

$$\begin{aligned}\left\|\frac{1}{(x+i)E_k}\right\|^2_{L^2(\mathbb{R})} &= \left\|\frac{\eta_{k-1}}{\eta_k}\cdot\frac{1}{(x+i)E_{k-1}}\right\|^2_{L^2(J_{k-1})} + \left\|\frac{1}{(x+i)E_k}\right\|^2_{L^2(\mathbb{R}\setminus J_{k-1})}\\ &\leq (1+2^{-k})^2\left\|\frac{1}{(x+i)E_{k-1}}\right\|^2_{L^2(\mathbb{R})} + 4\tau_{k-1}^{-2}.\end{aligned}$$

Since $4\tau_{k-1}^{-2} < 2^{-2k}$, one gets that

$$1 + \left\|\frac{1}{(x+i)E_k}\right\|^2_{L^2(\mathbb{R})} \leq (1+2^{-k})^2\left(1 + \left\|\frac{1}{(x+i)E_{k-1}}\right\|^2_{L^2(\mathbb{R})}\right).$$

This inequality also holds if $E_k = E_{k-1}$. Since $\prod_{k\geq 1}(1+2^{-k})^2$ converges and $\frac{1}{(x+i)E_0}$ is in $L^2(\mathbb{R})$, property (ii) follows.

Proof of (iii): It follows from (4.4) that there are constants C_k such that $\left\|\frac{1}{E_\ell}\right\|_{L^2(J_k)} \leq C_k$ for all ℓ. Now it is easy to get from the formulas $E_\ell = A\eta_\ell$ and (4.4) that for any fixed interval J_k, $\left\{\frac{1}{E_\ell}\right\}$ is a Cauchy sequence in $C(J_k)$. Since $\frac{1}{E_\ell}$ tend pointwise to $\frac{1}{E_*}$ on $\mathbb{R}$, (iii) follows. □

Example 4.3. 1. Let $t_n = n+\delta$, $n \in \mathbb{Z}$, $\delta \in (0,1)$. This spectrum may be annihilated by a one-dimensional perturbation, since we can take $E(z) = ie^{-\pi i(z-\delta)}$ or $E(z) = \sin\pi(z-\delta) + 2i\cos\pi(z-\delta)$.

2. The spectrum $\{t_n\} = \mathbb{N}$ is not removable, because $\mathbb{N}$ is not a zero set of a Cartwright class function.

Remark 4.4. If $\frac{1}{(z+i)E} \in H^2$ for a Hermite–Biehler function E, then, by Theorem 4.2, the spectrum $\{t_n\}$ (the zero set of $A = (E+E^*)/2$) is removable. A number of conditions in terms of the zeros of E ensuring this inclusion (which is equivalent to the fact that 1 is a function associated to $\mathcal{H}(E)$ in the terminology of [8]) have been obtained in [18, 4], while a criterion in terms of zeros of A and B was given in [40]. A slightly stronger property $1 \in \mathcal{H}(E)$ is closely related to the existence of positive minimal majorants for $\mathcal{H}(E)$ [17].

5. Conditions in terms of the generating function F. Proof of Theorem 1.1

It is possible to give a complete characterization of removable spectra solely in terms of the generating function F. We will need the following simple lemma about the Krein class $\mathcal{K}_1$.

Lemma 5.1. *Let $A(z) = v.p. \prod_n \left(1 - \frac{z}{t_n}\right)$ be a Cartwright class function with simple real zeros t_n. Then $A \in \mathcal{K}_1$ if and only if*

$$\sum_n \frac{1}{t_n^2 |A'(t_n)|} < +\infty. \tag{5.1}$$

Moreover, in this case in (3.2), *$R(z) \equiv R \equiv const$.*

Proof. We need to show only that (5.1) implies representation (3.2) with $R \equiv const$. Put

$$R(z) = \frac{1}{A(z)} - \sum_n \frac{1}{A'(t_n)} \Big(\frac{1}{z - t_n} + \frac{1}{t_n} \Big).$$

Obviously, R is an entire function. Moreover, since A is in the Cartwright class and $|A(iy)| \to \infty$ as $y \to \infty$, we conclude that $1/A$ is in the Smirnov class in the upper and in the lower half-planes. The function

$$\sum_n \frac{1}{A'(t_n)} \left(\frac{1}{z - t_n} + \frac{1}{t_n} \right) = z^2 \sum_n \frac{1}{t_n^2 A'(t_n)} \cdot \frac{1}{z - t_n} - z \sum_n \frac{1}{t_n^2 A'(t_n)}$$

is also in the Smirnov class (see Section 2.4). Thus R is of zero exponential type by Krein's theorem. Finally, note that $|R(iy)| = o(|y|)$, $|y| \to \infty$, and so R is a constant. □

Proof of Theorem 1.1. Assume that the spectrum $\{t_n\}$ is removable. Then there exist $E = A - iB$ and $\Theta = E^*/E$ as in Theorem 4.2, so that $1 + \Theta \notin H^2$. By (2.4), $\varphi = 1/E$ is of the form

$$\varphi(z) = \frac{1 + \Theta(z)}{2} \left(\varkappa + \sum_n c_n \Big(\frac{1}{t_n - z} - \frac{1}{t_n} \Big) \right),$$

where $c_n = a_n \overline{b}_n \mu_n$, and so $\sum_n t_n^{-2} |c_n| < \infty$. On the other hand,

$$\varphi = \frac{1}{E} = \frac{1 + \Theta}{2A},$$

and we conclude that $1/A$ has the representation of the form (1.4).

Conversely, if $A \in \mathcal{K}_1$, then A is of Cartwright class and

$$\frac{1}{A(z)} = q + \sum_n c_n \Big(\frac{1}{t_n - z} - \frac{1}{t_n} \Big), \qquad \sum_n \frac{|c_n|}{t_n^2} < \infty.$$

Now for any masses $\mu_n > 0$ we may choose a_n and b_n so that $c_n = a_n \overline{b}_n \mu_n$ and

$$\sum_n |a_n|^2 t_n^{-2} \mu_n < \infty, \qquad \sum_n |b_n|^2 t_n^{-2} \mu_n < \infty.$$

Indeed, note that $c_n = -1/A'(t_n) \neq 0$, and take $a_n = |c_n|^{1/2}\mu_n^{-1/2}$ and $b_n = \overline{c_n}|c_n|^{-1/2}\mu_n^{-1/2}$. Define Θ by formula (2.6) (with an arbitrary real constant δ). By construction, $1+\Theta \notin H^2$ (see the equivalence (2.5)). Then put

$$E = \frac{2A}{1+\Theta}.$$

Clearly, E is an entire function (the zeros sets of $1+\Theta$ and of A coincide) and

$$\frac{E^*(z)}{E(z)} = \frac{1+\Theta(z)}{1+\overline{\Theta(\overline{z})}} = \Theta(z) \tag{5.2}$$

since $\overline{\Theta(\overline{z})} = \big(\Theta(z)\big)^{-1}$. Thus, E is a Hermite–Biehler function and $A \notin \mathcal{H}(E)$ since $1+\Theta \notin H^2$. Finally, for the function $\varphi = 1/E$ we have

$$\varphi(z) = \frac{1+\Theta(z)}{2A(z)} = \frac{1+\Theta(z)}{2}\bigg(q + \sum_n c_n\Big(\frac{1}{t_n - z} - \frac{1}{t_n}\Big)\bigg).$$

We see that φ is of the form (2.2), whence, by Proposition 2.2, $\frac{1}{(z+i)E} = \frac{\varphi}{z+i} \in H^2$. So the spectrum $\{t_n\}$ is removable by Theorem 4.2. $\square$

A somewhat unexpected consequence of Theorem 1.1 is that adding a finite number of points to the spectrum helps it to become removable, while deleting a finite number of points may make it nonremovable.

Corollary 5.2. *(i) If $\{t_n\}$ is removable, then for any finite set $\{\tilde{t}_m\}_{m=1}^M$ disjoint with $\{t_n\}$ the spectrum $\{t_n\} \cup \{\tilde{t}_m\}_{m=1}^M$ is removable.*

(ii) If the spectrum $\{t_n\}$ is removable, then by deleting a finite number of elements of this sequence and adding the same number of other elements we will always obtain a removable spectrum.

Proof. The statements follows immediately from Theorem 1.1 since the multiplication by a polynomial maps the Krein class $\mathcal{K}_1$ into itself. $\square$

Corollary 5.3. *There exists a removable spectrum $\{t_n\}$, such that $\{t_n\}_{n\neq m}$ is nonremovable for any m.*

Proof. Clearly, the spectrum $\{t_n\}_{n\in\mathbb{Z}}$, where $t_n = n$ for $n \in \mathbb{Z}\setminus\{0\}$, and t_0 is any real noninteger number is removable (take $A(z) = \frac{(z-t_0)\sin\pi z}{z}$). Now consider the spectrum $\{t_n\} = \{n\}_{n\in\mathbb{Z}\setminus\{0\}}$. The corresponding generating function is $A(z) = \frac{\sin \pi z}{z}$ and $|F'(n)| \asymp |n|^{-1}$. Hence the series $\sum_{n\neq 0}\frac{1}{n^2|A'(t_n)|}$ diverges. Thus, $A \notin \mathcal{K}_1$ and so the spectrum is nonremovable. $\square$

6. Examples of removable and nonremovable spectra

In this subsection, we give some examples of removable and nonremovable spectra with power growth (one-sided and two-sided). To analyze the behavior of $|A'(t_n)|$ for the power growth of zeros we will use the Levin–Pfluger theory of functions of completely regular growth [24, Chapter 2]. Assume that t_n are all situated on the ray $\mathbb{R}_+$ and the counting function $n(r) = \#\{n : t_n \in [0,r]\}$ satisfies for some $\rho \in (0,1)$,

$$\lim_{r\to\infty}\frac{n(r)}{r^\rho} = D \in (0,\infty). \tag{6.1}$$

Assume also that, for some $d > 0$

$$t_{n+1} - t_n \ge d\, t_n^{1-\rho}. \tag{6.2}$$

Consider the discs $B_n = \{z : |z - t_n| < d\, t_n^{1-\rho}/2\}$. Then we have

$$\lim_{t\to+\infty,\, t\notin \cup_n B_n} \frac{\log|A(t)|}{t^\rho} = \pi D \cot \pi\rho \tag{6.3}$$

and

$$\lim_{t\to-\infty} \frac{\log|A(t)|}{|t|^\rho} = \frac{\pi D}{\sin \pi\rho}. \tag{6.4}$$

Moreover, it follows that

$$\frac{\log|A'(t_n)|}{t_n^\rho} \to \pi D\, \cot \pi\rho, \qquad n \to +\infty. \tag{6.5}$$

Example 6.1. 1. *Two-sided symmetric power growth.* Assume that for some $\rho \in (0,1)$, the spectrum $\{t_n\}$ satisfies

$$\lim_{r\to\infty} \frac{\#\{t_n \in (-r,0)\}}{r^\rho} = \lim_{r\to\infty} \frac{\#\{t_n \in (0,r)\}}{r^\rho} = D \in (0,\infty),$$

and $t_{n+1} - t_n \ge d|t_n|^{1-\rho}$, $d > 0$. Then $t_n \asymp C|n|^{\frac{1}{\rho}}$ as $|n| \to \infty$. It follows from (6.3)–(6.5) that

$$\log|A'(t_n)| \sim \pi D |t_n|^\rho \,\cot\frac{\pi\rho}{2} \asymp |n|, \qquad |n| \to +\infty,$$

and so, by Lemma 5.1, $A \in \mathcal{K}_1$ and the spectrum t_n is removable.

In particular, the spectrum $t_n = |n|^\gamma \operatorname{sign} n$, $n \in \mathbb{Z}\setminus\{0\}$, where t_0 is any nonzero number in $(0,1)$, is removable for any $\gamma > 1$ (and for $\gamma = 1$). Note also that if $\gamma < 1$, then the spectrum $\{t_n\}$ is not a zero set of a function of exponential type and, hence, is nonremovable.

2. *One-sided power growth.* Now let $t_n \in \mathbb{R}_+$ satisfy conditions (6.1)–(6.2). It follows from (6.5) and Lemma 5.1 that $\log|A'(t_n)| \asymp |t_n|^\rho$ and the spectrum $\{t_n\}$ is removable when $\rho < 1/2$, while for $\rho \in (1/2, 1)$ we have $\log|A'(t_n)| \asymp -|t_n|^\rho$ and the spectrum $\{t_n\}$ is nonremovable. In particular, the power spectra $t_n = n^\gamma$, $n \in \mathbb{N}$, are removable for $\gamma > 2$ and nonremovable for $\gamma < 2$.

3. *The limit case: square growth.* For one-sided power distributed zeros the limit case is the growth $t_n = n^2$, $n \in \mathbb{N}$. This situation is more subtle. In this case

$$A(z) = \prod_{n\in\mathbb{N}} \Big(1 - \frac{z}{n^2}\Big) = \frac{\sin(\pi\sqrt{z})}{\pi\sqrt{z}},$$

and so $|A'(t_n)| = (2t_n)^{-1} = (2n^2)^{-1}$. Then the series (5.1) converges and, by Lemma 5.1, the spectrum is removable. However, if we consider the spectrum $\{n^2\}_{n\ge 2}$, then the corresponding generating function A_1 satisfies $|A_1'(t_n)| \asymp |t_n|^{-2}$, and the spectrum is nonremovable.

4. *Two-sided nonsymmetric growth.* More generally, suppose that

$$\lim_{r\to+\infty} \frac{\#\{\pm t_n \in (0,r)\}}{r^{\rho_\pm}} = D_\pm \in (0,\infty),$$

and $t_{n+1} - t_n \ge d|t_n|^{1-\rho_\pm}$ for $\pm n > 0$, where $\rho_\pm \in (0,1)$ and $d > 0$. Define u_+, u_+ by

$$u_\pm = D_\pm \cot \pi\rho_\pm + \frac{D_\mp}{\sin \pi\rho_\mp}\,.$$

Then the same arguments as above imply that the spectrum is removable if both u_- and u_+ are positive and is not removable if at least one of these numbers is negative. In particular, if $\rho_-, \rho_+ < 1/2$, then the spectrum is removable.

Remark 6.2. 1. The special role of the exponent 2 in the power distributed spectra is well known; it may be seen, e.g., in the problems of weighted polynomial approximation on discrete subsets of $\mathbb{R}$ [7].

2. In the study of power growth, the regularity of the sequence is important. It is easy to see that for any $\gamma > 2$ there exists a subset of $\{n^\gamma\}_{n\in\mathbb{N}}$, which is nonremovable (take the set n^γ, $n \in [m_k, m_k + l_k]$ for appropriately chosen m_k, $l_k \to \infty$).

Example 6.3. Let $a > 0$ and consider two shifted progressions:

$$t_n = \begin{cases} n + a, & n \ge 0,\ n \in \mathbb{Z}, \\ n + 1 - a, & n < 0,\ n \in \mathbb{Z}, \end{cases}$$

that is, $\{t_n\} = \{\dots, -a-1, -a\} \cup \{a, a+1, \dots\}$. Then

$$A(z) = \prod_{n=0}^{\infty}\Big(1 - \frac{z^2}{(n+a)^2}\Big) = \frac{\Gamma(a)^2}{\Gamma(a+z)\Gamma(a-z)} = \frac{\Gamma(a)^2}{\pi\Gamma(a+z)} \sin\pi(a-z)\Gamma(1-a-z)\,.$$

Therefore for positive $k \in \mathbb{Z}$,

$$|A'(k+a)| = \frac{\Gamma(a)^2}{\Gamma(k+2a)}\,\Gamma(k+1) \asymp \Gamma(a)^2 k^{1-2a} \quad \text{as } k \to +\infty.$$

Since A is an even function, the series $\sum_{k>0} \frac{1}{k^2|A'(t_k)|}$ converges (and the spectrum is removable) if and only if $a < 1$.

Example 6.4. One more class of examples with a nonremovable spectrum can be obtained if we take a sequence of pairs of close points. In this case the spectrum is 'almost multiple' and thus nonremovable. Let $\{t_n\}$ be a separated sequence (i.e., $\inf_n(t_{n+1} - t_n) > 0$) and consider the set $\{t_n\} \cup \{t_n + \delta_n\}$, where $\delta_n \to 0$. If δ_n are sufficiently small, then we can achieve that $|A'(t_n)|$ be small and, thus, (5.1) is not satisfied.

7. Livšic's theorem on dissipative Volterra operators with one-dimensional imaginary part

In this section we use our model to prove the above-mentioned theorem of Livšic; it says that any *dissipative Volterra operator*, which is a rank one perturbation of a selfadjoint operator, is unitary equivalent to the integration operator ([26] or [13, Ch. I, Th. 8.1]) [1]. Namely, we show the following:

[1] We express our gratitude to N. Nikolski who attracted our attention to Livšic's theorem and suggested to deduce it using our methods.

Theorem 7.1 (Livšic, [26]). *Let $\mathcal{L}_0 = \mathcal{A}_0 + i\mathcal{B}_0$ be a dissipative Volterra operator (in some Hilbert space H) such that both $\mathcal{A}_0$ and $\mathcal{B}_0$ are selfadjoint and $\mathcal{B}_0$ is of rank one. Then the spectrum of $\mathcal{A}_0$ is given by $s_n = c(n+1/2)^{-1}$, $n \in \mathbb{Z}$, for some $c \in \mathbb{R}$, $c \neq 0$.*

From this, one may deduce that $\mathcal{A}_0$ is unitary equivalent to the selfadjoint integral operator (having the same spectrum)

$$(\tilde{\mathcal{A}}f)(x) = i\int_0^{2\pi c} f(t)\,\mathrm{sign}\,(x-t)\,dt, \qquad f \in L^2(0, 2\pi c),$$

while $\mathcal{L}_0$ is unitary equivalent to the integration operator $(\tilde{\mathcal{L}}f)(x) = 2i\int_0^x f(t)\,dt$.

Since $\mathcal{B}_0 \geq 0$, we have $\mathcal{B}_0 x = (x, b_0)b_0$ for some $b_0 \in H$. By passing to the unbounded inverses, we obtain (after an obvious unitary equivalence) a singular rank one perturbation $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ of the operator $\mathcal{A}$ of multiplication by the independent variable in some space $L^2(\mu)$, where $\mu = \sum_n \mu_n \delta_{t_n}$, $t_n = s_n^{-1}$. Moreover, in the case of the positive imaginary part, we may assume that $\varkappa = -1$ and $a = ib$.

Applying the functional model from Section 2, we construct a pair (Θ, φ) as in Theorem 2.1. Let $E = A - iB \in HB$ be such that $\Theta = E^*/E$ and let $g = \varphi E$. Then by (2.2), (2.3), we have

$$\frac{B(z)}{A(z)} = \delta + \sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n}\right)|b_n|^2 \mu_n,$$

$$\frac{g(z)}{A(z)} = -1 + i\sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n}\right)|b_n|^2 \mu_n,$$

whence $g = -A + i(B - \delta A)$.

Since $\mathcal{L}$ (and, thus, the model operator $\mathcal{T}$) is the inverse to a Volterra operator, the spectrum of $\mathcal{T}$ is the point at infinity. By Lemma 2.3, g has no zeros in $\mathbb{C}$. Also, by Theorem 4.2, the function E is of Carthwright class, and the same is true for g. We conclude that $g(z) = \exp(i\pi cz)$ for some real c. Thus,

$$e^{i\pi cz} = -A(z) + i\big(B(z) - \delta A(z)\big).$$

The functions A and B are real on the real axis. Taking the real parts, we get $A(z) = -\cos \pi cz$, and so $t_n = c^{-1}(n+1/2)$, $n \in \mathbb{Z}$, as required.

8. Volterra rank one perturbations generated by 'smooth' vectors

Let $\mathcal{A}_0$ be a compact selfadjoint operator with the simple point spectrum $\{s_n\}$, that is, the operator of multiplication by x in $L^2(\nu)$, where $\nu = \sum_n \nu_n \delta_{s_n}$. In this section we show that, for a rank one perturbation $\mathcal{L}_0 = \mathcal{A}_0 + ab^*$, the property of being a Volterra operator is compatible with a certain smoothness of the vectors a and b. On the other hand, recall that the classical completeness theorem of Macaev [30] (see also [12, Chapter V]) states that in the case when a or b is in the range of $\mathcal{A}_0$ (i.e., $a \in xL^2(\nu)$ or $b \in xL^2(\nu)$) and $\mathrm{Ker}\,\mathcal{L}_0 = 0$, the perturbed operator $\mathcal{L}_0$ has a complete set of eigenvectors (or root vectors in case of multiple spectrum):

Theorem (Macaev, 1961). *If $\mathcal{L}_0 = \mathcal{A}_0(I+S)$, where $\mathcal{A}_0$, S are compact operators on a Hilbert space, $\mathcal{A}_0$ is selfadjoint and S is in the Macaev ideal $\mathfrak{S}_\omega$ (i.e., its singular numbers s_k satisfy $\sum_{k\geq 1}\frac{s_k}{k} < \infty$) and $\ker \mathcal{A}_0 = \ker(I+S) = 0$, then $\mathcal{L}_0$ and $\mathcal{L}_0^*$ have complete sets of eigenvectors.*

The following theorem shows that any weaker smoothness of a and b can be achieved for Volterra rank one perturbations. A special case of this result was given in [5, Theorem 0.6].

Theorem 8.1. *Let $s_n \to 0$, $s_n \neq 0$, and assume that $\{t_n\}$, where $t_n = s_n^{-1}$, is a removable spectrum. Let A be the corresponding function in the Krein class $\mathcal{K}_1$, given by (4.1). Assume that for some $\gamma \in (0, 2)$ we have*

$$\sum_n \frac{1}{|t_n|^\gamma |A'(t_n)|} < \infty.$$

Let $\mathcal{A}_0$ be a selfadjoint operator with the point spectrum $\{s_n\}$ and trivial kernel (i.e., $\mathcal{A}_0$ is the operator of multiplication by x in $L^2(\nu)$ where $\nu = \sum_n \nu_n \delta_{s_n}$). Then for any $\alpha_1, \alpha_2 \in [0, 1)$ with $\alpha_1 + \alpha_2 \leq 2 - \gamma$ there exist $a \in |x|^{\alpha_1} L^2(\nu)$ and $b \in |x|^{\alpha_2} L^2(\nu)$ such that $a, b \notin L^2(\nu)$ and the spectrum of the perturbed operator $\mathcal{L}_0 = \mathcal{A}_0 + ab^*$ equals $\{0\}$.

Proof. Let us pass to the equivalent problem for a singular perturbation of an unbounded operator $\mathcal{A}$ (which is unitary equivalent to $\mathcal{A}_0^{-1}$) on $L^2(\mu)$, where $\mu = \sum_n \mu_n \delta_{t_n}$, $t_n = s_n^{-1}$, and $a'_n = (\mathcal{A}^{-1} a)_n = a_n / s_n$, $b'_n = b_n / s_n$. Thus, for any α_1 and α_2 as above, we need to find $\mu = \sum_n \mu_n \delta_{t_n}$, $\varkappa \in \mathbb{R}$, and $a', b' \notin L^2(\mu)$ such that

$$\sum_n |a'_n|^2 |t_n|^{2\alpha_1 - 2} \mu_n < \infty, \qquad \sum_n |b'_n|^2 |t_n|^{2\alpha_2 - 2} \mu_n < \infty \tag{8.1}$$

(note that $a'_n = a_n t_n$) and the function

$$\varphi(z) = \frac{1 + \Theta(z)}{2} \cdot \left(\varkappa + \sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n} \right) a'_n \overline{b'_n} \mu_n \right) \tag{8.2}$$

has no zeros in $\mathbb{C}$.

For a function $A \in \mathcal{K}_1$ we have

$$\frac{1}{A(z)} = q + \sum_n c_n \left(\frac{1}{t_n - z} - \frac{1}{t_n} \right), \tag{8.3}$$

where $q = 1/A(0)$, $c_n = -1/A'(t_n)$ and $\sum t_n^{-2} |c_n| < \infty$. We represent c_n as $c_n = a'_n \overline{b'_n} \mu_n$, where a'_n and b'_n have the required properties. Once such a' and b' have been constructed, we define the function Θ by the formulas (2.2) and (2.4) with b'_n in place of b_n, that is, we put

$$i \frac{1 - \Theta(z)}{1 + \Theta(z)} = \sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n} \right) |b'_n|^2 \mu_n.$$

Next, define φ by (8.2), with $a'_n \overline{b'_n} \mu_n = c_n$ and with q in place of $\varkappa$. If we now put $E = \frac{2A}{1+\Theta}$, then, clearly, E is an entire function, $\varphi = 1/E$ by (8.3), and $\Theta = E^*/E$ (see (5.2)) whence $E \in HB$. Thus, the function φ has no zeros.

Let $\{\mu_n\}$ be an arbitrary sequence of positive numbers. Put

$$a'_n = \frac{|c_n|^{1/2} |t_n|^{(2 - 2\alpha_1 - \gamma)/2}}{\mu_n^{1/2}}, \qquad b'_n = \frac{|c_n|^{1/2} |t_n|^{(2\alpha_1 + \gamma - 2)/2}}{\mu_n^{1/2}}.$$

Then

$$\sum_n |a'_n|^2 |t_n|^{2\alpha_1 - 2} \mu_n = \sum_n \frac{|c_n|}{|t_n|^\gamma} < \infty,$$

$$\sum_n |b'_n|^2 |t_n|^{2\alpha_2-2}\mu_n = \sum_n \frac{|c_n|}{|t_n|^{4-2\alpha_1-2\alpha_2-\gamma}} < \infty,$$

since $\alpha_1 + \alpha_2 \le 2 - \gamma$.

If a' and b' are not in $L^2(\mu)$, then the theorem is proved. Otherwise, choose a sequence $p_{2n+1} \ge 1$ such that

$$\sum_n p_{2n+1}^2 |a'_{2n+1}|^2 |t_{2n+1}|^{2\alpha_1-2}\mu_{2n+1} < \infty, \qquad \sum_n p_{2n+1}^2 |a'_{2n+1}|^2 \mu_{2n+1} = \infty$$

(this is, obviously, possible, because $|t_n| \to \infty$ and $\alpha_1 < 1$). Analogously, we choose $p_{2n} \le 1$ so that

$$\sum_n p_{2n}^{-2} |b'_{2n}|^2 |t_{2n}|^{2\alpha_2-2}\mu_{2n} < \infty, \qquad \sum_n p_{2n}^{-2} |b'_{2n}|^2 \mu_{2n} = \infty.$$

Then, clearly, $\tilde a_n = p_n a'_n$ and $\tilde b_n = p_n^{-1} b'_n$ are not in $L^2(\mu)$, $\tilde a \in x^{\alpha_1-1}L^2(\mu)$, $\tilde b \in x^{\alpha_2-1}L^2(\mu)$ and $\tilde a_n \overline{\tilde b_n} = c_n$. □

Remark 8.2. One can compare Theorem 8.1 with Macaev's theorem mentioned above as well as with the following result from [5] (Theorem 3.3, Statement (2)): *Let $\mathcal{A}$ be an unbounded cyclic selfadjoint operator with discrete spectrum $\{t_n\}$, $t_n \ne 0$, and let the data $(a, b, \varkappa)$ satisfy*

$$\sum_n \frac{|a_n b_n|\mu_n}{|t_n|} < \infty, \tag{8.4}$$

$$\sum_n \frac{a_n \overline{b}_n \mu_n}{t_n} \ne \varkappa. \tag{8.5}$$

Then the singular rank one perturbation $\mathcal{L} = \mathcal{L}(\mathcal{A}, a, b, \varkappa)$ and its adjoint $\mathcal{L}^*$ have complete sets of eigenvectors.

Note that (8.4) is satisfied if $1 \le \alpha_1 + \alpha_2$. However, there is no contradiction with Theorem 8.1. Indeed, looking at the the asymptotics when $y \to \infty$ in

$$\frac{1}{A(iy)} = \varkappa - \sum_n \frac{c_n}{t_n} + \sum_n \frac{c_n}{t_n - iy},$$

where $c_n = a_n \overline{b}_n \mu_n$, we see that (8.5) is not satisfied for the perturbation constructed in Theorem 8.1.

Thus, in contrast to Macaev's theorem (which applies to the so-called weak perturbations of the form $\mathcal{A}_0(I+S)$ or $(I+S)\mathcal{A}_0$), we have the following corollary of Theorem 8.1:

Corollary 8.3. *For any $\alpha_1, \alpha_2 \in (0,1)$, $\alpha_1 + \alpha_2 > 1$, there exist a positive compact operator $\mathcal{A}_0$ and its rank one perturbation $\mathcal{L}_0$ of the form*

$$\mathcal{L}_0 = \mathcal{A}_0 + \mathcal{A}_0^{\alpha_1} S \mathcal{A}_0^{\alpha_2},$$

where S is a rank one operator and $\operatorname{Ker}\mathcal{L}_0 = \operatorname{Ker}\mathcal{L}_0^* = 0$, *such that $\mathcal{L}_0$ is a Volterra operator.*

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